Academic Theories Meet The Practice of Active Portfolio Management

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Shingo Goto, Ph.D.[1]

1 Review Questions

1.1 Arbitrage Free Pricing Basics

Assume that the rates of return on three assets ($i = 1, 2, 3$) are described by two factors (say, "value" and "growth" factors, $\tilde{F}_v$ and $\tilde{F}_g$):

$$\begin{aligned} \tilde{r}_1 - r_f &= \mu_1 + \beta_{1,v}\tilde{F}_v + \beta_{1,g}\tilde{F}_g, \\ \tilde{r}_2 - r_f &= \mu_2 + \beta_{2,v}\tilde{F}_v + \beta_{2,g}\tilde{F}_g, \\ \tilde{r}_3 - r_f &= \mu_3 + \beta_{3,v}\tilde{F}_v + \beta_{3,g}\tilde{F}_g, \end{aligned}$$

The two factors, $\tilde{F}_v$ and $\tilde{F}_g$, are random variables with mean zero. (They are already demeaned.) r_f is the risk-free rate of return. Note that $\mu_i \equiv E[\tilde{r}_i] - r_f$ ($i = 1, 2, 3$) denote expected excess returns on the three assets. Why should a linear pricing rule holds (e.g., $\mu_i = \beta_{i,v}\lambda_v + \beta_{i,g}\lambda_g$ for $i = 1, 2, 3$) if no arbitrage opportunities exist? Please explain (briefly).

Hint: Consider a fully invested portfolio of the three assets, (w_1, w_2, w_3) such that $\sum_{i=1}^{3} w_i = 1$. The portfolio's expected excess return is

$$E\left[\tilde{r}_p\right] - r_f = \sum_{i=1}^{3} w_i\mu_i + (\sum_{i=1}^{3} w_i\beta_{i,v})\tilde{F}_v + (\sum_{i=1}^{3} w_i\beta_{i,g})\tilde{F}_g.$$

We can choose a riskless portfolio by setting $(\sum_{i=1}^{3} w_i\beta_{i,v}) = 0$ and $(\sum_{i=1}^{3} w_i\beta_{i,g}) = 0$. The riskless portfolio should have an expected return of r_f. to preclude arbitrage (Expected excess return of the riskless portfolio must be zero.)

<u>Solution</u>

Construct a risk-free (zero-beta) portfolio $w = (w_1, w_2, w_3)'$ such that

$$\begin{aligned} w_1\beta_{1,v\backslash} + w_2\beta_{2,v} + w_3\beta_{3,v} &= 0 \\ w_1\beta_{1,g} + w_2\beta_{2,g} + w_3\beta_{3,g} &= 0 \end{aligned}$$

[1] Please let me know at shingo_goto@uri.edu if you detect or suspect typos. Thank you.

To preclude arbitrage, this zero-beta portfolio must have an expected return equal to r_f (expected excess return must be zero). Hence the intercept term is zero:

$$w_1\mu_1 + w_2\mu_2 + w_3\mu_3 = 0.$$

We can summarize these conditions as

$$\underbrace{\begin{bmatrix} \mu_1 & \mu_2 & \mu_3 \\ \beta_{1,v} & \beta_{2,v} & \beta_{3,v} \\ \beta_{1,g} & \beta_{2,g} & \beta_{3,g} \end{bmatrix}}_{A} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since $w \neq 0$ ($\sum_{i=1}^{3} w_i = 1$), the matrix A must be singular. (One can show that no arbitrage exists if and only if $\det(A) = 0$). That is, a row of A is a linear combination of the other 2 rows. Thus there must exist λ_v, λ_g ($\lambda_v \neq 0$ or/and $\lambda_g \neq 0$) such that

$$\begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = \begin{bmatrix} \beta_{1,v} \\ \beta_{2,v} \\ \beta_{3,v} \end{bmatrix} \lambda_v + \begin{bmatrix} \beta_{1,g} \\ \beta_{2,g} \\ \beta_{3,g} \end{bmatrix} \lambda_g.$$

That is, we have obtained a linear pricing rule in the absence of arbitrage opportunities:

$$E[r_i] - r_f = \beta_{i,v}\lambda_v + \beta_{i,g}\lambda_2;\ i = 1, 2, 3.$$

We can naturally extend this logic to the case of $N > 3$ assets.

1.2 Mean Variance Analysis, No Arbitrage, and Beta Pricing

(1) Review Questions

There are N risky assets and a risk-free asset in the economy. Assume that the following two-factor return generating process holds:

$$\tilde{r}_i - r_f = \mu_i + \beta_{i,1}\tilde{F}_1 + \beta_{i,2}\tilde{F}_2 + \tilde{\epsilon}_i;\ i = 1, ..., N,$$

where r_f is the risk-free rate and $\tilde{\epsilon}_i$ is the idiosyncratic return of the i-th asset ($i = 1, ..., N$). For simplicity, let's assume that the two factors are already demeaned and orthogonalized, i.e., $E[\tilde{F}_1] = E[\tilde{F}_2] = 0$ and $E[\tilde{F}_1\tilde{F}_2] = 0$ ($Cov[\tilde{F}_1, \tilde{F}_2] = 0$). The variances of the two factors are $Var[\tilde{F}_1] = \sigma_1^2$ and $Var[\tilde{F}_2] = \sigma_2^2$. We consider fully-invested portfolios (i.e., portfolio weights of the N risky assets sum to one).

1. Please show that an exact beta pricing relation (e.g., $E[\tilde{r}_i] - r_f \equiv \mu_i = \beta_{i,1}\lambda_1 + \beta_{i,2}\lambda_2$) obtains when a well-diversified portfolio with only factor risk (i.e., without any idiosyncratic risk) is mean-variance efficient.

Hint 1: Excess return of this "efficient" portfolio can be expressed as

$$\tilde{r}_e - r_f = \mu_e + \beta_{e,1}\tilde{F}_1 + \beta_{e,2}\tilde{F}_2. \tag{1}$$

Hint 2: The relationship between the expected return on "any" portfolio p (that is not necessarily on the frontier) and a frontier portfolio e (other than the Global Minimum Variance Portfolio) can be stated as

$$Cov\left[r_p, r_e\right] = \psi E\left[r_p\right] + \zeta.$$

Proof: We can show this by recognizing that a mean-variance efficient portfolio (w_e), that solves the optimization problem (with Lagrange multipliers ψ and ζ)

$$\min_w L = \frac{1}{2}w'\Sigma w + \psi(E\left[r_p\right] - w'\bar{R}) + \zeta(1 - w'\mathbf{1}),$$

can be expressed as

$$w_e = \psi\Sigma^{-1}\boldsymbol{\mu} + \zeta\Sigma^{-1}\mathbf{1},$$

where $\boldsymbol{\mu} = (E[\tilde{r}_1], ..., E[\tilde{r}_N])'$ is the $N \times 1$ vector of expected returns and $\mathbf{1}$ is the $N \times 1$ vector of ones. Let w_p denote the vector of portfolio weights of p. It then follows that

$$\begin{aligned} Cov\left[r_p, r_e\right] &= w_p'\Sigma w_e = w_p'\Sigma\left(\psi\Sigma^{-1}\boldsymbol{\mu} + \zeta\Sigma^{-1}\mathbf{1}\right) \\ &= \psi w_p'\boldsymbol{\mu} + \zeta w_p'\mathbf{1} = \psi E\left[r_p\right] + \zeta. \end{aligned}$$

2. Consider well-diversified "pure factor portfolios" whose excess returns are described as:

$$\begin{aligned} \tilde{r}_{F1} - r_f &= \lambda_1 + \tilde{F}_1, \\ \tilde{r}_{F2} - r_f &= \lambda_2 + \tilde{F}_2. \end{aligned}$$

Expected excess returns on these pure factor portfolios are the factor risk premiums, i.e., $E\left[\tilde{r}_{F1}\right] - r_f \equiv \lambda_1$ and $E\left[\tilde{r}_{F2}\right] - r_f \equiv \lambda_2$. The covariance matrix of the two factor portfolio returns is a diagonal matrix (with diagonal elements σ_1^2 and σ_2^2) because the two factors are orthogonal to each other. What is the highest Sharpe ratio one can achieve by combining these two factor portfolios?

Hint: The tangency portfolio (that achieves the highest Sharpe ratio) is $w = k \times V^{-1}\boldsymbol{\mu}$ where V is the covariance matrix and $\boldsymbol{\mu}$ is the vector of expected excess returns. k is a scaling constant ($k = \frac{1}{\mathbf{1}'V^{-1}\boldsymbol{\mu}}$).

Solution to the 1st part

Following the hint, let's use the expression (1) for the excess return of the mean variance efficient portfolio. Then, for any assets $i = 1, ..., N$, we have the following relation must hold:

$$Cov\left[r_i, r_e\right] = \psi E\left[r_i\right] + \zeta = \psi(E\left[r_i\right] + \frac{\zeta}{\psi}).$$

For the risk-free asset, $Cov\left[r_f, r_e\right] = 0$ (because r_f is not random) and hence $\psi r_f + \zeta = 0 \Leftrightarrow r_f = -\frac{\zeta}{\psi}$. It follows that, for any asset, $i = 1, ..., N$,

$$\begin{aligned} E\left[r_i\right] - r_f &= \frac{1}{\psi} Cov\left[r_i, r_e\right] \\ &= \frac{1}{\psi} Cov\left[\beta_{i,1}\tilde{F}_1 + \beta_{i,2}\tilde{F}_2 + \tilde{\epsilon}_i, \beta_{e,1}\tilde{F}_1 + \beta_{e,2}\tilde{F}_2\right] \\ &= \frac{1}{\psi}\beta_{i,1}\underbrace{\beta_{e,1}\sigma_1^2}_{\lambda_1} + \frac{1}{\psi}\beta_{i,2}\underbrace{\beta_{e,2}\sigma_2^2}_{\lambda_2}. \end{aligned}$$

By setting $\lambda_1 \equiv \frac{1}{\psi}\beta_{e,1}\sigma_1^2$ and $\lambda_2 \equiv \frac{1}{\psi}\beta_{e,2}\sigma_2^2$, we obtain the exact pricing rule, $E\left[r_i\right] - r_f = \beta_{i,1}\lambda_1 + \beta_{i,2}\lambda_2$, $\forall i = 1, ..., N$.

Solution to the 2nd part

For the tangency portfolio $w_\tau = k \times V^{-1}\boldsymbol{\mu}$, the expected excess return is $w_\tau'\boldsymbol{\mu} = k \times \boldsymbol{\mu}'V^{-1}\boldsymbol{\mu}$. Its ex ante volatility is $\sqrt{w_\tau' V w_\tau} = k \times \sqrt{\boldsymbol{\mu}'V^{-1}\boldsymbol{\mu}}$. Thus the ex ante Sharpe ratio of the tangency portfolio is $\frac{w_\tau'\boldsymbol{\mu}}{\sqrt{w_\tau' V w_\tau}} = \sqrt{\boldsymbol{\mu}'V^{-1}\boldsymbol{\mu}}$, where

$$\boldsymbol{\mu}'V^{-1}\boldsymbol{\mu} = \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}^{-1} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \frac{\lambda_1^2}{\sigma_1^2} + \frac{\lambda_2^2}{\sigma_2^2}.$$

That is, the highest Sharpe ratio one can achieve from the two factor portfolios is $\sqrt{\frac{\lambda_1^2}{\sigma_1^2} + \frac{\lambda_2^2}{\sigma_2^2}}$. Note that the squared Sharpe ratio of the tangency portfolio is equal to the sum of the squared sharpe ratios of the two factor portfolios.

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(2) No Arbitrage and Beta Pricing: A Note (Not Questions)

Beta Pricing with the Stochastic Discount Factor No arbitrage implies the existence of the stochastic discount factor (SDF) $\tilde{m} > 0$ that satisfies

$$E\left[\tilde{m}\left(\tilde{r}_i - r_f\right)\right] = 0 \text{ or } E\left[\tilde{m}(1 + \tilde{r}_i)\right] = 1$$

for $i = 1, ..., N$. ($E[\tilde{m}] = \frac{1}{1+r_f}$.)

$$E\left[\tilde{m}\left(\tilde{r}_i - r_f\right)\right] = 0$$

$$\implies E\left[\tilde{m}(\mu_i + \beta_{i,1}\tilde{F}_1 + \beta_{i,2}\tilde{F}_2 + \tilde{\epsilon}_i)\right] = 0$$
$$\implies E[\tilde{m}]\mu_i + \beta_{i,1}\underbrace{E[\tilde{m}\tilde{F}_1]}_{Cov[\tilde{m},\tilde{F}_1]} + \beta_{i,2}\underbrace{E[\tilde{m}\tilde{F}_2]}_{Cov[\tilde{m},\tilde{F}_2]} + \underbrace{E[\tilde{m}\tilde{\epsilon}_i]}_{Cov[\tilde{m},\tilde{\epsilon}_i]} = 0$$
$$\implies \mu_i = \beta_{i,1}\lambda_1 + \beta_{i,2}\lambda_{1,2} + \alpha_i$$

where $\lambda_1 \equiv -\frac{Cov[\tilde{m},\tilde{F}_1]}{E[\tilde{m}]}$ and $\lambda_2 \equiv -\frac{Cov[\tilde{m},\tilde{F}_2]}{E[\tilde{m}]}$ are the two factor risk premiums. $\alpha_i = -\frac{Cov[\tilde{m},\tilde{\epsilon}_i]}{E[\tilde{m}]}$ is the pricing error. For a well-diversified portfolio (with small $Var[\tilde{\epsilon}_p]$), a linear pricing rule $\mu_i = \beta_{i,1}\lambda_1 + \beta_{i,2}\lambda_{1,2}$ holds well, though there could be some risk premiums associated with the idiosyncratic risk of individual assets.

Risk Neutral Pricing Appealing to No Arbitrage, we can also apply the risk-neutral pricing principle:

$$E^Q[\tilde{r}_i] = r_f \text{ for } i = 1, ..., N.$$

$$\implies E^Q\left[\mu_i + \beta_{i,1}\tilde{F}_1 + \beta_{i,2}\tilde{F}_2 + \tilde{\epsilon}_i\right] = 0$$
$$\implies \mu_i + \beta_{1,i}E^Q\left[\tilde{F}_1\right] + \beta_{i,2}E^Q\left[\tilde{F}_2\right] + E^Q[\tilde{\epsilon}_i] = 0.$$

where $E^Q[.]$ is the mean under the risk-neutral probability measure. Defining $\lambda_1 \equiv -E^Q\left[\tilde{F}_1\right]$, $\lambda_2 \equiv -E^Q\left[\tilde{F}_2\right]$, and $\alpha_i \equiv -E^Q[\tilde{\epsilon}_i]$, we obtain

$$\mu_i = \beta_{1,i}\lambda_1 + \beta_{2,i}\lambda_2 + \alpha_i.$$

The pricing error α_i is the mean of the residual return in the risk-neutral world, $E^Q[\tilde{\epsilon}_i]$. (Note that $E^Q[\tilde{x}] = \frac{E[\tilde{m}\tilde{x}]}{E[\tilde{m}]}$ for a random variable $\tilde{x}$.)

2 For Discussion: Multifactor Beta Pricing Models

2.1 Setup

Consider an investment universe with a risk-free asset, N risky assets, and K factor portfolios. Returns of the N risky assets are generated by a K-factor model ($K < N$):

$$\tilde{r}_i - E[\tilde{r}_i] = \beta_{i,1}\tilde{f}_1 + \cdots + \beta_{i,K}\tilde{f}_K + \tilde{\varepsilon}_i,\ i = 1, ..., N.$$

Let us summarize this in a vector form:

$$\tilde{R} - E\left[\tilde{R}\right] = B\tilde{F} + \tilde{\epsilon}, \tag{2}$$

where $\tilde{R} \equiv (\tilde{r}_1, ..., \tilde{r}_N)'$ and $\tilde{\epsilon} = (\tilde{\varepsilon}_1, ..., \tilde{\varepsilon}_N)'$ are $N \times 1$ vectors, $E\left[\tilde{R}\right]$ is the $N \times 1$ vector of mean returns, and $\tilde{F} = \left(\tilde{f}_1, ..., \tilde{f}_K\right)'$ is the $K \times 1$ factor vector. B is the $N \times K$ matrix of factor loadings (betas) whose (i, k)th element is $\beta_{i,k}$, ($i = 1, ..., N$; $k = 1, ..., K$). We use r_f to denote the risk-free rate.

- The factor model (2) says that unexpected returns are attributed to the effects of the K factors and the residual component.

- The factors satisfy $E[\tilde{F}] = 0$ and $E[\tilde{F}\tilde{F}'] = \Phi$ (Φ is the $K \times K$ factor covariance matrix). In general, the factors are correlated with each other.

- $\tilde{F}$ can be viewed as unexpected returns of the K factor portfolios. That is,

$$\tilde{F} = \tilde{R}_{Factor} - E[\tilde{R}_{Factor}].$$

  where $\tilde{R}_{Factor}$ is the $K \times 1$ vector of factor portfolio returns.

  Remarks Factor portfolios are not constructed only from the N assets. (For example, the factor portfolios are formed from a larger universe that encompasses the N assets under consideration.) We need this assumption to ensure that the factor portfolios are not redundant.

- The residual return vector $\tilde{\epsilon}$ has mean zero and covariance matrix Σ, i.e., $E[\tilde{\epsilon}] = 0$, $E[\tilde{\epsilon}\tilde{\epsilon}'] = \Sigma$. Σ may or may not be diagonal.

- V denotes the covariance matrix of $\tilde{R}$. Our setup implies a "risk model" of the form:

$$V = B\Phi B' + \Sigma.$$

2.2 Question: Mean Variance Analysis and Exact Beta Pricing

Suppose that we can form a mean-variance efficient portfolio from a linear combination of the K factor portfolios. In this case, we can show that an exact beta pricing model holds. Let's write the exact beta pricing relation as

$$E\left[\tilde{R}\right] - \mathbf{1}r_f = B\lambda, \tag{3}$$

where $\mathbf{1}$ is the $N \times 1$ vector of ones. What is the maximum (ex ante) Sharpe ratio you can achieve when you can invest in both the N risk assets and the K factor portfolios?

Solution

Since a portfolio of the factor portfolios (only) is mean-variance efficient, we can consider the tangency portfolio of the factor portfolios (that achieves the highest Sharpe ratio). The maximum Sharpe ratio is $\sqrt{\lambda'\Phi^{-1}\lambda}$.

2.3 Question: Pricing Errors

We now consider the case where we do not have the exact factor pricing, so the mean-variance efficient portfolio cannot be formed from a linear combination of the K factor portfolios.

Suppose that the expected excess return of the N risky assets is expressed as

$$E\left[\tilde{r}\right] - \mathbf{1}r_f = \alpha + B\lambda,$$

where α is an $N \times 1$ vector of "pricing errors" (or "alphas"). The pricing error is cross-sectionally orthogonal to the factor betas, in the sense that $\lim_{N\to\infty} \alpha'\Sigma^{-1}B = 0$. (Let's assume that N is sufficiently large and we can assume $\alpha'\Sigma^{-1}B = 0$.)

Question Let SR_{Factor} be the maximum Sharpe ratio you have just obtained under the exact beta pricing model [expression (3)]. Please show that the (ex ante) maximum Sharpe ratio (SR_{Max}) satisfies

$$SR_{Max}^2 = SR_{Factor}^2 + \alpha'\Sigma^{-1}\alpha.$$

That is

$$SR_{Max}^2 = SR_{Factor}^2 + IR_{Max}^2.$$

Notes

- $IR_{Max} \equiv \sqrt{\alpha'\Sigma^{-1}\alpha}$ is the maximum "Information Ratio" (IR) one can achieve in this setup. IR_{Max}^2 capture the potential value created by active portfolio management. When Σ is diagonal, $IR_{Max}^2 = \sum_{i=1}^{N}(\frac{\alpha_i}{\sigma_{\varepsilon_i}})^2$. This result is a generalization of the classic Treynor-Black (1973)[2] framework.

- We can connect this theoretical result to popular asset pricing tests such as Gibbons, Ross, and Shanken's (1989)[3] (GRS) test. These tests typically examine the significance of statistics of the form:

$$\begin{aligned} J_{Wald} &= \alpha' \left[Var[\alpha]\right]^{-1} \alpha = T\frac{IR_{Max}^2}{\left(1 + SR_{Factor}^2\right)} \sim \chi_N^2 \\ J_{GRS} &= \frac{T-N-1}{N}\frac{IR_{Max}^2}{\left(1 + SR_{Factor}^2\right)} \sim F_{N,T-N-1} \end{aligned}$$

Solution

[2] Treynor, J.L. and Black, F.(1973), "How to Use Security Analysis to Improve Portfolio Selection," Journal of Business, 46(1), pp.66-86.

[3] Gibbons, M.R., Ross, S.A., and Shanken, J. (1989), "A Test of the Efficiency of a Given Portfolio," Econometrica, 57(5), pp.1121-1152.

The covariance matrix of $N+K$ factor returns is

$$\Omega = \begin{bmatrix} B\Phi B' + \Sigma & B\Phi \\ \Phi B' & \Phi \end{bmatrix}$$

Using the formula (for the inverse of a partitioned matrix, please see the appendix), Ω^{-1} simplifies to

$$\Omega^{-1} = \begin{bmatrix} \Sigma^{-1} & -\Sigma^{-1}B \\ -B'\Sigma^{-1} & \Phi^{-1} + B'\Sigma^{-1}B \end{bmatrix}.$$

The maximum Sharpe ratio squared, SR^2_{Max}, is

$$\begin{aligned} SR^2_{Max} &= \begin{bmatrix} \alpha' + \lambda' B' & \lambda' \end{bmatrix} \begin{bmatrix} \Sigma^{-1} & -\Sigma^{-1}B \\ -B'\Sigma^{-1} & \Phi^{-1} + B'\Sigma^{-1}B \end{bmatrix} \begin{bmatrix} \alpha + B\lambda \\ \lambda \end{bmatrix} \\ &= \text{(a fun manipulation that greatly simplifies the expression)} \\ &= \underbrace{\alpha'\Sigma^{-1}\alpha}_{IR^2_{Max}} + \underbrace{\lambda'\Phi^{-1}\lambda'}_{SR^2_{Factor}}. \end{aligned}$$

2.4 Question: Active Management

Suppose we are able to uncover the source of the pricing error (or "alpha") α. We have found that α is linearly related to a "neglected factor exposure" (or a "signal"), $Z = (z_1, ..., z_N)'$, that we can observe at the beginning of the period. The elements of Z $(z_1, ..., z_N)$ are cross-sectionally independent and identically distributed with mean zero and variance σ_z^2, i.e., $\frac{1}{N}Z'Z = \sigma_z^2$ as $N \to \infty$. We assume that N is sufficiently large so that we can assume $Z'Z = N\sigma_z^2$. σ_z is a measure of cross-sectional dispersion of the signal.

With the discovery of Z, we can express and asset/portfolio's excess return over the benchmark factor portfolio return as

$$\begin{aligned} \tilde{R} - B \cdot \tilde{R}_{Factor} &= \alpha + \tilde{\epsilon} \\ &= Z\gamma + \tilde{\epsilon}, \end{aligned}$$

where Z and $\tilde{\epsilon}$ are orthogonal to each other. Being a neglected factor, Z may also help explain covariances among residual returns. We can express the residual covariance matrix $E[\tilde{\epsilon}\tilde{\epsilon}'] = \Sigma$ as

$$\Sigma = \eta^2 ZZ' + \Delta.$$

To simplify the following discussion, we assume $\Delta = \delta^2 I$, where δ^2 is the variance of idiosyncratic returns. (Idiosyncratic returns are uncorrelated with each other.) We can view η^2 as the variance of the neglected factor return. That is, when we express $\tilde{\varepsilon} = Z\tilde{h} + \tilde{u}$, $E[\tilde{h}] = E[\tilde{u}] = 0$, and $\eta^2 \equiv Var[\tilde{h}]$ and $E[\tilde{u}\tilde{u}'] \equiv \Delta = \delta^2 I$.

By observing Z at the beginning of the period, we are able to tell which assets are "undervalued" (likely to outperform) and which assets are "overvalued" (likely to under-perform) relative to the benchmark beta pricing model [equation (3)]. We assume that no constraints or frictions inhibit our trading activities (i.e., there are no "limits of arbitrage," no transaction costs, etc.).

Questions Please consider the following questions for discussion.

1. How would you design a zero-cost portfolio ("arbitrage portfolio") that exploits the knowledge of $\alpha = Z\gamma$ to maximize the Sharpe ratio?
2. What is the (ex ante) maximum Sharpe ratio you can thus achieve? Does it increase without bound as we increase the "breadth" $N \to \infty$? Could we give an interpretation of the "Fundamental Law of Active Management" à la Grinold (1989)[4] along this line?

Solution to the 1st part

To maximize the IR, one can form the arbitrage portfolio in the form of $w_a \propto V^{-1}Z$, where $\propto$ means "proportional to." When Z and all column vectors of B are orthogonal to each other in the sense that $B'\Sigma^{-1}Z = 0$, we can also express w_a as $w_a \propto \Sigma^{-1}Z$. This is because, by the Woodbury identity (please see the appendix),

$$\begin{aligned} \Sigma V^{-1} Z &= \Sigma\Sigma^{-1}Z - \Sigma\Sigma^{-1}B\left(\Phi^{-1} + B'\Sigma^{-1}B\right)^{-1}\underbrace{B'\Sigma^{-1}Z}_{=0} = Z \\ &\Leftrightarrow\ V^{-1}Z = \Sigma^{-1}Z. \end{aligned}$$

Solution to the 2nd part

The maximum IR squared is $IR^2_{Max} = \alpha'\Sigma^{-1}\alpha$. Using the Sherman-Morrison formula (please see the appendix),

$$\begin{aligned} IR^2_{Max} &= \alpha'\Sigma^{-1}\alpha \\ &= \gamma^2 Z'\left(\eta^2 ZZ' + \Delta\right)^{-1} Z \\ &= \gamma^2 Z'\left[\Delta^{-1} - \frac{\eta^2\Delta^{-1}ZZ'\Delta^{-1}}{1+\eta^2 Z'\Delta^{-1}Z}\right] Z \\ &= \gamma^2\left(Z'\Delta^{-1}Z\right) - \gamma^2\eta^2\frac{\left(Z'\Delta^{-1}Z\right)^2}{1+\eta^2\left(Z'\Delta^{-1}Z\right)}. \end{aligned}$$

Let $x \equiv \left(Z'\Delta^{-1}Z\right) > 0$. We can then express IR^2_{Max} as

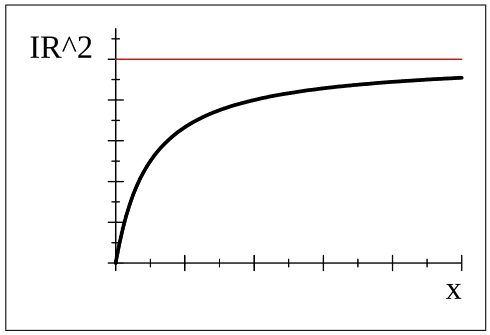
$$IR^2_{Max} = \gamma^2 x - \frac{\gamma^2\eta^2 x^2}{1+\eta^2 x} = \frac{\gamma^2}{\frac{1}{x}+\eta^2}.$$

[4] Grinold, R.C. (1989), "The Fundamental Law of Active Management," Journal of Portfolio Management, 15(3), pp.30-37.

But,

$$x = \delta^{-2} Z'Z = N\frac{\sigma_z^2}{\delta^2}.$$

$x \to \infty$ as $N \to \infty$. Assuming that η^2 is positive, IR^2_{Max} is small when x is small (because the denominator gets large). IR^2_{Max} increases with x, but it does not increase without bound. In fact, $IR^2_{Max} \to \frac{\gamma^2}{\eta^2}$ (the squared IR of the neglected factor) as $N \to \infty$.



The red line corresponds to $IR^2 = \frac{\gamma^2}{\eta^2}$.

However, when $\eta = 0$, that is, when the signal is purely idiosyncratic, IR^2_{Max} can increase without bound.

Notes: Relation with the Grinold-Kahn (1999)[5] Framework The Fundamental Law of Active Management is

$$IR = IC \times \sqrt{N}$$

where IC is the "information coefficient." In our context, IC is the cross-sectional correlation between Z and $\tilde{R} - B\tilde{R}_{Factor} = Z\gamma + \tilde{\epsilon}$. When $\eta = 0$ (i.e., the signal Z is completely firm specific) and for sufficiently large N,

$$IC = \frac{Cov[Z, Z\gamma + \tilde{\epsilon}]}{\sqrt{Var[Z]}\sqrt{Var[\tilde{\epsilon}]}} = \frac{\sigma_z^2 \gamma}{\sigma_z \delta} = \frac{\sigma_z \gamma}{\delta}.$$

By replacing γ in our alpha forecast $\alpha = Z\gamma$ with IC, we have an alternative expression of the alpha forecast:

$$\alpha = Z\gamma = \underset{\text{volatility}}{\delta} \times IC \times \underset{\text{score}}{\frac{Z}{\sigma_z}}.$$

This is the popular "$alpha = Volatility \times IC \times Score$" recipe for active portfolio management. Recall that we have assumed $\eta = 0$ – that is, we have assumed that Z is not a source of return covariances in this derivation.

[5] Grinold, R.C. and Kahn, R.N. (1999), Active Portfolio Management: A Quantitative Approach to Providing Superior Returns and Controlling Risk, 2nd edition, McGraw-Hill.

APPENDIX

A Some Useful Formulas for Portfolio Management

A.1 The Inverse of a Partitioned Matrix

Let the $(M \times M)$ matrix A be partitioned into sub-matrices so that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where A, A_{11}, and A_{22} are nonsingular. Then, the inverse of A is

$$\begin{aligned} A^{-1} &= \begin{bmatrix} D_{11} & -D_{11}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}D_{11} & A_{22}^{-1} + A_{22}^{-1}A_{21}D_{11}A_{12}A_{22}^{-1} \end{bmatrix} \\ &= \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}D_{22}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}D_{22} \\ -D_{22}A_{21}A_{11}^{-1} & D_{22} \end{bmatrix} \end{aligned}$$

where

$$\begin{aligned} D_{11} &= \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1} \\ D_{22} &= \left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1} \end{aligned}$$

We can also express D_{11} and D_{22} as

$$\begin{aligned} D_{11} &= \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1} = A_{11}^{-1} + A_{11}^{-1}A_{12}D_{22}A_{21}A_{11}^{-1} \\ &= A_{11}^{-1} + A_{11}^{-1}A_{12}\left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1}A_{21}A_{11}^{-1} \end{aligned}$$

and

$$\begin{aligned} D_{22} &= \left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1} = A_{22}^{-1} + A_{22}^{-1}A_{21}D_{11}A_{12}A_{22}^{-1} \\ &= A_{22}^{-1} + A_{22}^{-1}A_{21}\left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1}A_{12}A_{22}^{-1} \end{aligned}$$

Personal Notes on The Inversion of a Partitioned Covariance Matrix

We use this formula mostly for inverting a partitioned covariance matrix. Let Σ be the covariance matrix of $(\tilde{x}', \tilde{y}')'$ (where $\tilde{x}$ and $\tilde{y}$ are random vectors are independently and identically distributed) where

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \sim IID\left(\mu, \Sigma\right)$$

$$\mu = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}, \ \Sigma \equiv \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{bmatrix}.$$

Although it is very difficult for me to memorize the formula for Σ^{-1}, I would use the following steps to calculate and interpret Σ^{-1}.

1. Consider the following regressions ($\tilde{x}$ on $\tilde{y}$ and $\tilde{y}$ on $\tilde{x}$):

$$\begin{aligned}\tilde{x} &= a_{x|y} + B_{x|y}\tilde{y} + \tilde{\varepsilon}_{x|y}, \\ \tilde{y} &= a_{y|x} + B_{y|x}\tilde{x} + \tilde{\varepsilon}_{y|x},\end{aligned}$$

   where $B_{x|y} \equiv \Sigma_{xy}\Sigma_{yy}^{-1}$, $B_{y|x} \equiv \Sigma_{yx}\Sigma_{xx}^{-1}$, $a_{x|y} = \mu_x - B_{x|y}\mu_y$ and $a_{y|x} = \mu_y - B_{y|x}\mu_x$.

2. Then, we can express Σ^{-1} as

$$\Sigma^{-1} = \begin{bmatrix} Var\left[\tilde{\varepsilon}_{x|y}\right]^{-1} & -Var\left[\tilde{\varepsilon}_{x|y}\right]^{-1} B_{x|y} \\ -Var\left[\tilde{\varepsilon}_{y|x}\right]^{-1} B_{y|x} & Var\left[\tilde{\varepsilon}_{y|x}\right]^{-1} \end{bmatrix}$$

   where $Var\left[\tilde{\varepsilon}_{x|y}\right]$ and $Var\left[\tilde{\varepsilon}_{y|x}\right]$ are the residual variances.

$$\begin{aligned}Var\left[\tilde{\varepsilon}_{x|y}\right] &= \Sigma_{xx} - B_{x|y}\Sigma_{yy}B'_{x|y}. \\ Var\left[\tilde{\varepsilon}_{y|x}\right] &= \Sigma_{yy} - B_{y|x}\Sigma_{xx}B'_{y|x}.\end{aligned}$$

We can also re-express Σ^{-1} as

$$\Sigma^{-1} = \begin{bmatrix} Var\left[\tilde{\varepsilon}_{x|y}\right]^{-1} & 0 \\ 0 & Var\left[\tilde{\varepsilon}_{y|x}\right]^{-1} \end{bmatrix} \begin{bmatrix} I & -B_{x|y} \\ -B_{y|x} & I \end{bmatrix}$$

I have personally found this decomposition very useful [e.g. Stevens (1998)[6]]. See Goto and Xu (2015)[7] for an application.

Suppose $\tilde{x}$ and $\tilde{y}$ are active portfolio returns (with zero exposures to usual factors). To maximize the Information Ratio (IR), we choose a portfolio

$$\begin{aligned}\begin{bmatrix} w_x \\ w_y \end{bmatrix} &= c \times \Sigma^{-1} \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix} \\ &= c \times \begin{bmatrix} Var\left[\tilde{\varepsilon}_{x|y}\right]^{-1} & 0 \\ 0 & Var\left[\tilde{\varepsilon}_{y|x}\right]^{-1} \end{bmatrix} \begin{bmatrix} \mu_x - B_{x|y}\mu_y \\ \mu_y - B_{y|x}\mu_x \end{bmatrix} \\ &= c \times \begin{bmatrix} Var\left[\tilde{\varepsilon}_{x|y}\right]^{-1} a_{x|y} \\ Var\left[\tilde{\varepsilon}_{y|x}\right]^{-1} a_{y|x} \end{bmatrix}\end{aligned}$$

where c is a scaling constant.

[6] Stevens, Guy V.G. (1998), "On the Inverse of the Covariance Matrix in Portfolio Analysis," Journal of Finance 53(5), 1821-1827.

[7] Goto, S. and Xu, Y. (2015), "Improving Mean Variance Optimization through Sparse Hedging Restrictions," Journal of Financial and Quantitative Analysis, 50(6), pp.1415-1441.

A.2 Sherman-Morrison-Woodbury Matrix Identity

The Woodbury Formula

When A and C are nonsingular, the Woodbury matrix identity (the matrix inversion lemma) is

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B\left(C^{-1} + DA^{-1}B\right)^{-1} DA^{-1}.$$

Consider a direct application to a risk model, $V = B\Phi B' + \Sigma$

$$\begin{aligned} V^{-1} &= \left(B\Phi B' + \Sigma\right)^{-1} \\ &= \Sigma^{-1} - \Sigma^{-1}B\left(\Phi^{-1} + B'\Sigma^{-1}B\right)^{-1} B'\Sigma^{-1}. \end{aligned} \tag{4}$$

In practice, the following expression has implementation advantages over equation (4) when we need to deal with singular (or near singular) Φ.

$$V^{-1} = \Sigma^{-1} - \Sigma^{-1}B\left(\Phi B'\Sigma^{-1}B + I\right)^{-1} \Phi B'\Sigma^{-1}$$

The Sherman-Morrison Formula

A special case of the Woodbury matrix identity is the Sherman-Morrison formula:

$$\left(A + uv'\right)^{-1} = A^{-1} - \frac{A^{-1}uv'A^{-1}}{1 + v'A^{-1}u}$$

where u and v are column vectors and $1 + v'A^{-1}u \neq 0$.