Dimensional Analysis, Leverage Neutrality, and Market Microstructure Invariance

Albert S. Kyle

University of Maryland

Anna A. Obizhaeva

New Economic School

Minsk, Belarus
August 26, 2017

Overview

This paper combines **dimensional analysis**, **leverage neutrality**, and a principle of **market microstructure invariance** to derive scaling laws.

- Scaling laws relate transaction costs functions, bid-ask spreads, bet sizes, number of bets, and other financial variables in terms of dollar trading volume and volatility.
- These laws are tested using a data set of trades in the Russian and U.S. stock markets and find a strong support in the data.
- These scaling laws provide useful metrics for risk managers and traders; scientific benchmarks for evaluating issues related to high frequency trading, market crashes, and liquidity measurement; and guidelines for designing policies.

A General Picture

The scaling laws in finance can be derived using different approaches:

- "Market Microstructure Invariance: Empirical Hypotheses" (Ecma, 2016): Empirical conjectures and tests.
- "Market Microstructure Invariance: A Dynamic Equilibrium Model": Dynamic equilibrium model of speculative trading in which liquidity constrained investors seek to profit from trading on signals with invariant cost.
- This paper: Physicists' approach, apply dimensional analysis (consistency of units, Buckingham π-theorem)

Dimensional Analysis

Physics researchers obtain powerful results by using dimensional analysis to reduce the dimensionality of problems (the size and number of molecules in a mole of gas, the size of the explosive energy, turbulence).

- **Physics:** fundamental units of mass, distance, and time & conservation laws based on laws of physics.
- **Finance:** fundamental units of time, currency, and shares & conservation laws based on no-arbitrage restrictions.

Oscillation of a Pendulum?

Suppose the time of oscillation T of a pendulum $T = f(M, L, g)$. It has units of seconds $[s]$.

- *mass* M is in $[kg]$,
- *length* L is in $[m]$,
- *gravity acceleration* g is in $[m/s^2]$.

Buckingham π theorem claims that rescaled T is a function of $N-3$ rescaled dimensionless variables

$$T = \sqrt{\frac{L}{g}} \cdot f(\text{dimensionless variables}) = \sqrt{\frac{L}{g}} \cdot f(\cdot) = \sqrt{\frac{L}{g}} \cdot const.$$

Then, use the law of conservation of energy to find a constant 2π.

If $T = f(M, L, g, x_1, x_2)$, then $T = \sqrt{\frac{L}{g}} \cdot f(\text{scaled x1, scaled x2})$.

How Big was the Bomb?

The first atomic blast, the Trinity Test in New Mexico in 1945, had an explosive yield of about 20 kilotons, but this value was secret.

Based on photographs of the Trinity Test released by the US Army in 1947 and dimensional analysis, Taylor guessed the size E from

$$R = \left(\frac{Et^2}{\rho}\right)^{1/5},$$

where R is radius, E is energy, t is time, ρ is density of air.

Dimensional Analysis and Finance

In financial markets, institutional investors trade by implementing speculative "bets" which move prices. A bet is a decision to buy or sell a quantity of institutional size.

Trading is costly; bets tend to move market prices.

Dimensional Analysis and Finance

$$\begin{aligned}
\text{Price} &= P_{jt} = 40.00 \text{ dollars/share} \\
\text{Trading Volume} &= V_{jt} = 1.00 \text{ million shares/day} \\
\text{Volatility} &= \sigma_{jt}^2 = 0.02^2/\text{day} \\
\text{Tick Size} &= K_{\text{MIN}} = 0.01 \text{ dollars/share} \\
\text{Minimum Lot Size} &= Q_{\text{MIN}} = 100 \text{ shares} \\
\text{Size of Bet} &= Q_{jt} = 10\,000 \text{ shares} \\
\text{Execution Horizon} &= H_{jt} = 1 \text{ day} \\
\text{Trade Size} &= X_{jt} = 200 \text{ shares} \\
\text{Number of Trades} &= N_{jt} = 5000/\text{day} \\
\text{Quoted Bid-Ask Spread} &= S_{jt} = 0.02 \text{ dollars/share} \\
\text{Market Impact Cost} &= G_{jt} = 40 \times 10^{-4} = 40 \text{ basis points?} \\
\text{Average ``Bet Cost''} &= C = 1600 \text{ dollars}
\end{aligned}$$

Transaction Costs

Let G_{jt} denote the price impact cost as a fraction of the value traded $Q_{jt} \cdot P_{jt}$. The price impact G_{jt} is dimensionless, e.g. in basis points, and it is a function of

$$G_{jt} := g(Q_{jt}, P_{jt}, V_{jt}, \sigma^2_{jt}, C).$$

- bet size Q_{jt} in units of shares,
- stock price P_{jt} in units of dollars per share,
- share volume V_{jt} in units of shares-per-day,
- volatility σ^2_{jt} in units of per-day,
- bet cost C in units of dollars.

Dimensional Analysis

Since the value of $G_{jt} := g(Q_{jt}, P_{jt}, V_{jt}, \sigma^2_{jt}, C)$ is dimensionless, consistency of units implies that it cannot depend on the dimensional quantities P_{jt}, Q_{jt}, and σ^2_{jt}.

Thus, *dimensional analysis* implies that the function $g()$ can be further simplified by writing it as function of two dimensionless variables.

$$\begin{aligned} G_{jt} &= P^0_{jt} \cdot Q^0_{jt} \cdot (\sigma^2_{jt})^0 \cdot f(\text{two dimensionless variables}) \\ &= f(\text{two dimensionless variables}). \end{aligned}$$

Dimensional Analysis

There are three sets of distinct units and five dimensional quantities—Q_{jt}, P_{jt}, V_{jt}, σ_{jt}^2, C.

Form two independent dimensionless quantities:

$$L_{jt} := \left(\frac{m^2 \cdot P_{jt} \cdot V_{jt}}{\sigma_{jt}^2 \cdot C} \right)^{\alpha}, \qquad Z_{jt} := \frac{P_{jt} \cdot Q_{jt}}{L_{jt} \cdot C},$$

where m^2 is a dimensionless scaling constant.

Thus, *dimensional analysis* implies that the function g can be further simplified by writing it as $g(L_{jt}, Z_{jt})$.

$$G_{jt} := g(L_{jt}, Z_{jt}).$$

Leverage Neutrality

Introduce a conservation law in the form of *leverage neutrality*: The cost of exchanging cash is zero. Therefore adding cash or riskless debt does not affect the cost G_{jt} of exchanging a risking asset.

If $(A-1)P$ dollars of cash or debt is added to P_{jt}, then

$$\begin{aligned} P_{jt} &\to P_{jt} \cdot A & Q_{jt} &\to Q_{jt} \\ \sigma_{jt}^2 &\to \sigma_{jt}^2 \cdot A^{-2} & V_{jt} &\to V_{jt} \\ L_{jt} &\to L_{jt} \cdot A^{3\alpha} & C &\to C \\ Z_{jt} &\to Z_{jt} & G_{jt} &\to G_{jt} \cdot A^{-1} \end{aligned}$$

The change in G_{jt} keeps dollar transaction costs $G_{jt} \cdot Q_{jt} \cdot P_{jt}$ constant. Now choose α so that $1/L_{jt}$ has the same leverage scaling as G_{jt}. This implies $\alpha = 1/3$:

Leverage Neutrality

Percentage cost G_{jt} of executing a bet of Q_{jt} shares changes by a factor A^{-1}, since dollar cost did not change but dollar value changed. Leverage neutrality implies that

$$g(A \cdot L_{jt}, Z_{jt}) = A^{-1} \cdot g(L_{jt}, Z_{jt}).$$

If $A = L_{jt}^{-1}$, then $g(L_{jt}, Z_{jt}) = L_{jt}^{-1} \cdot g(1, Z_{jt})$.

Define $f(Z_{jt}) := g(1, Z_{jt})$ and get a very **important formula**:

$$G_{jt} = \frac{1}{L_{jt}} \cdot f(Z_{jt}).$$

Transaction Costs Model

A general specification for transaction costs functions consistent with the scaling implied by dimensional analysis and leverage neutrality:

$$g(Q_{jt}, P_{jt}, V_{jt}, \sigma_{jt}^2, C) = \left(\frac{\sigma_{jt}^2 \cdot C}{m^2 \cdot P_{jt} \cdot V_{jt}}\right)^{\mathbf{1/3}} \cdot f\left(\left(\frac{\sigma_{jt}^2 \cdot C}{m^2 \cdot P_{jt} \cdot V_{jt}}\right)^{\mathbf{1/3}} \cdot \frac{P_{jt} \cdot Q_{jt}}{C}\right).$$

It is consistent with different assumptions about the shape of the function f.

Liquid Markets

Scale m and C so that

$$E\{|Z_{jt}|\} = 1 \qquad \text{and} \qquad C = E\{|G_{jt}|P_{jt}Q_{jt}|\}.$$

The variables L_{jt} and Z_{jt} have an intuitive interpretation:

- $\frac{1}{L_{jt}} = \frac{C}{E\{P_{jt} \cdot |\tilde{Q}_{jt}|\}}$ is **"illiquidity index"** measuring average cost.
- $Z_{jt} = \frac{P_{jt} \cdot \tilde{Q}_{jt}}{E\{P_{jt} \cdot |\tilde{Q}_{jt}|\}}$ is **"scaled bet size"** relative to the average size.
- $m = \frac{E\{|Q_{jt}|\}}{\left(E\{Q_{jt}^2\}\right)^{1/2}}$ is moment ratio.

Liquid Markets

More liquid markets are associated with more bets of larger sizes (2-to-1 ratio):

- Bet size $E\{P_{jt} \cdot |\tilde{Q}_{jt}|\} = C \cdot L_{jt}$.
- Number of bets per day $\gamma_{jt} = \frac{\sigma_{jt}^2}{m^2} \cdot L_{jt}^2$.

Market Microstructure Invariance

Extra assumptions are necessary to make our predictions operational.

- Three of the quantities—asset price P_{jt}, trading volume V_{jt}, and return volatility σ_{jt}—can be observed directly or readily estimated from public data feeds.
- Q_{jt} is a characteristic of a bet privately known to a trader.
- **Invariance:** the dollar value of C and the dimensionless scaling parameter m^2 are the same!

These assumptions are related to bet size and transaction costs invariance hypotheses. Preliminary calibration gives $C \approx \$2,000$ and $m^2 \approx 0.25$.

Transaction Costs Models

Suppose f is a power function of the form $f(Z_{jt}) = \bar{\lambda} \cdot |Z_{jt}|^{\omega}$.

- A proportional bid-ask spread cost ($\omega = 0$) implies

$$G_{jt} = \mathrm{const} \cdot \frac{1}{L_{jt}}.$$

- A linear market impact cost ($\omega = 1$) implies

$$G_{jt} = \mathrm{const} \cdot \frac{P_{jt} \cdot |Q_{jt}|}{C \cdot L_{jt}^2}.$$

- A square-root market impact cost ($\omega = 1/2$) implies

$$G_{jt} = \mathrm{const} \cdot \sigma_{jt} \cdot \left(\frac{|Q_{jt}|}{V_{jt}} \right)^{1/2}.$$

Liquidity

Our measure of liquidity is consistent in terms of units:

$$L_{jt} := \left(\frac{m^2 \cdot P_{jt} \cdot V_{jt}}{\sigma_{jt}^2 \cdot C} \right)^{1/3} \sim \left(\frac{P_{jt} \cdot V_{jt}}{\sigma_{jt}^2} \right)^{1/3} .$$

It is the correct way to construct empirical measure of Kyle's lambda.

Russian Data

- One-minute data from the Moscow Exchange for January–December 2015 provided by Interfax Ltd.
- 50 Russian stocks in the RTS index as of June 15, 2015.
- The Russian stock market is centralized with all trading implemented in a consolidated limit-order book.
- Small tick and lot sizes.

U.S. Data

- One-minute data from the Trades and Quotes (TAQ) dataset for January–December 2015.
- 500 U.S. stocks in the S&P 500 index as of June 15, 2015.
- The U.S. stock market is fragmented, and securities are traded simultaneously at dozens of exchanges.
- Tick size of one cent, and lot sizes of 100 shares.

Tests for Bid-Ask Spread

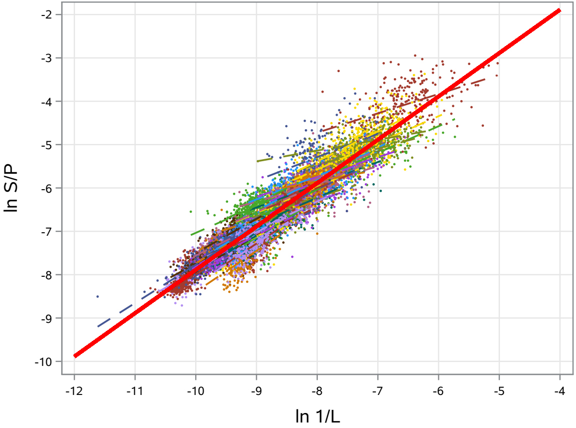
Let S_{jt} denote the bid-ask spread. Since

$$\frac{S_{jt}}{P_{jt}} = const \cdot \frac{1}{L_{jt}},$$

we get

$$\log\left(\frac{S_{jt}}{P_{jt}}\right) = \mathrm{const} + \mathbf{1} \cdot \log\left(\frac{1}{L_{jt}}\right).$$

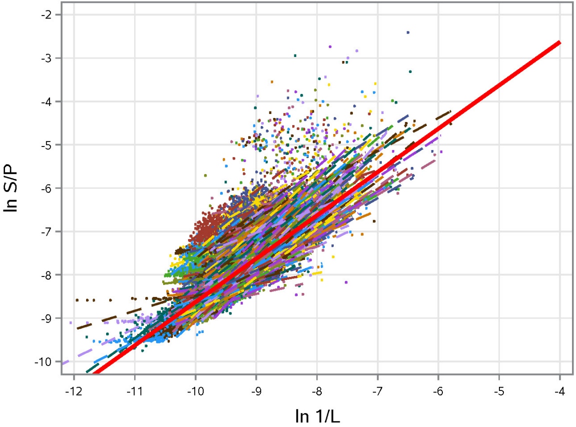
Spread: Results for Russian Data



In aggregate sample, the slope is close to 1! R-square is 0.876.

$$\log(S_{jt}/P_{jt}) = 2.093 + \mathbf{0.998} \cdot \log(1/L_{jt})$$

Spread: Results for U.S. Data



In aggregate sample, the slope is close to 1! R-square is 0.450.

$$\log(S_{jt}/P_{jt}) = 1.011 + \mathbf{0.961} \cdot \log(1/L_{jt})$$

Tests for Number of Trades

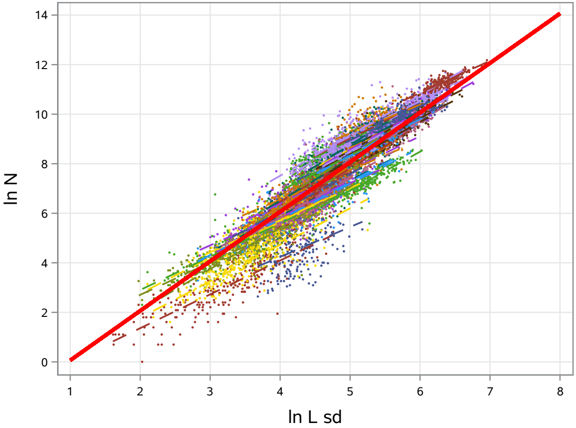
Let N_{jt} denote the number of trades and suppose $N_{jt} \sim \gamma_{jt}$. Since

$$\gamma_{jt} = \frac{\sigma_{jt}^2}{m^2} \cdot L_{jt}^2,$$

we get

$$\log(N_{jt}) = \mathrm{const} + \mathbf{2} \cdot \log(\sigma_{jt} L_{jt}).$$

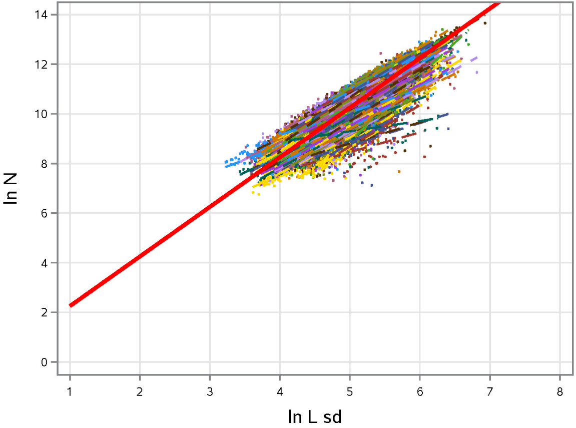
Number of Trades: Results for Russian Data



In aggregate sample, the slope is close to 2! R-square is 0.882.

$$\log(N_{jt}) = -3.085 + \mathbf{2.239} \cdot \log(\sigma_{jt} L_{jt})$$

Number of Trades: Results for U.S. Data



In aggregate sample, the slope is close to 2! R-square is 0.702.

$$\log(N_{jt}) = 1.005 + \mathbf{1.842} \cdot \log(\sigma_{jt} L_{jt})$$

Extensions

The empirical implications of dimensional analysis, leverage invariance, and market microstructure invariance can be generalized to incorporate various trading **frictions**.

Generalized Transaction Costs Formula

Add the **execution horizon** T_{jt} (in units of time), the **tick size** K_{jt}^{MIN} (in dollars per share), and the **lot size** Q_{jt}^{MIN} (in shares).

Re-scale variables to make them dimensionless and leverage neutral using the four variables P_{jt}, V_{jt}, σ_{jt}^2, and C:

- $\frac{|Q_{jt}|}{T_{jt}} \to \frac{|Q_{jt}|}{V_{jt} \cdot T_{jt}}$,
- $K_{jt}^{MIN} \to K_{jt}^{MIN} \cdot \frac{L_{jt}}{P_{jt}}$,
- $Q_{jt}^{MIN} \to Q_{jt}^{MIN} \cdot \frac{\sigma_{jt}^2 \cdot L_{jt}^2}{V_{jt}}$.

Generalized Transaction Costs Formula

$$G_{jt} = \frac{1}{L_{jt}} \cdot f\left(\frac{P_{jt} \cdot Q_{jt}}{C \cdot L_{jt}}, \frac{|Q_{jt}|}{V_{jt} \cdot T_{jt}}, \frac{K_{jt}^{MIN} \cdot L_{jt}}{P_{jt}}, \frac{Q_{jt}^{MIN} \cdot \sigma_{jt}^2 \cdot L_{jt}^2}{V_{jt}}\right).$$

Optimal Execution Horizon

Suppose the optimal execution horizon T_{jt}^* for an order of Q_{jt} shares depends on P_{jt}, V_{jt}, σ_{jt}^2, C, K_{jt}^{MIN}, and Q_{jt}^{MIN}.

Since $|Q_{jt}|/(V_{jt} \cdot T_{jt}^*)$ is dimensionless and leverage neutral, the same logic implies:

$$\frac{|Q_{jt}|}{V_{jt} \cdot T_{jt}^*} = h^* \left(\frac{P_{jt} \cdot Q_{jt}}{C \cdot L_{jt}}, \frac{K_{jt}^{MIN} \cdot L_{jt}}{P_{jt}}, \frac{Q_{jt}^{MIN} \cdot \sigma_{jt}^2 \cdot L_{jt}^2}{V_{jt}} \right).$$

If tick size and lot size do not affect execution horizon, $|Q_{jt}|/(V_{jt} \cdot T_{jt}^*)$ depends only on $Z_{jt} := P_{jt} \cdot Q_{jt}/(C \cdot L_{jt})$.

Optimal Tick Size and Lot Size

Setting optimal tick size and minimum lot size is of interest for exchange officials and regulators.

Let K_{jt}^{MIN*} and Q_{jt}^{MIN*} denote optimal tick size and optimal minimum lot size, respectively.

Optimal Tick Size and Lot Size

Since the scaled optimal quantities $K_{jt}^{MIN*} \cdot L_{jt}/P_{jt}$ and $Q_{jt}^{MIN*} \cdot L_{jt}^2 \cdot \sigma_{jt}^2/V_{jt}$ are dimensionless and leverage neutral, the scaling laws for these market frictions are

$$K_{jt}^{MIN*} = \mathrm{const} \cdot \frac{P_{jt}}{L_{jt}}, \qquad Q_{jt}^{MIN*} = \mathrm{const} \cdot \frac{V_{jt}}{L_{jt}^2 \cdot \sigma_{jt}^2}.$$

General Formula for Bid-Ask Spread

Here is a formula for bid-ask spread for the market with frictions:

$$\frac{S_{jt}}{P_{jt}} = \frac{1}{L_{jt}} \cdot s \left(\frac{K_{jt}^{MIN} \cdot L_{jt}}{P_{jt}}, \frac{Q_{jt}^{MIN} \cdot \sigma_{jt}^2 \cdot L_{jt}^2}{V_{jt}} \right).$$

If tick size and minimum lot size have no influence on quoted bid-ask spreads, then the the relationship simplifies to $S_{jt}/P_{jt} \sim 1/L_{jt}$.

General Formula for Trading Patterns

Here are general formulas for trade sizes $\tilde{X}_{jt}$ and number of trades N_{jt}:

$$\mathrm{Prob}\left\{\frac{P_{jt}\cdot\tilde{X}_{jt}}{C\cdot L_{jt}} < z\right\} = F^{Q}_{jt}\left(z, \frac{K^{MIN}_{jt}\cdot L_{jt}}{P_{jt}}, \frac{Q^{MIN}_{jt}\cdot\sigma^2_{jt}\cdot L^2_{jt}}{V_{jt}}\right).$$

$$N_{jt} = \sigma^2_{jt}\cdot L^2_{jt}\cdot f\left(\frac{K^{MIN}_{jt}\cdot L_{jt}}{P_{jt}}, \frac{Q^{MIN}_{jt}\cdot\sigma^2_{jt}\cdot L^2_{jt}}{V_{jt}}\right).$$

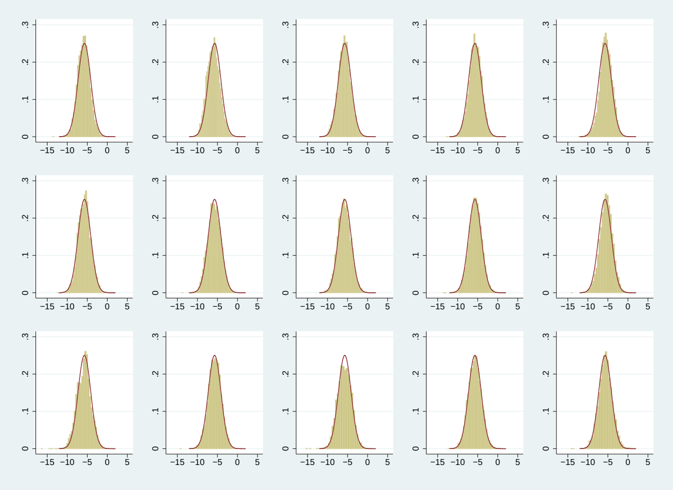
Conclusions

There is a growing empirical evidence that the scaling laws discussed above match patterns in financial data, at least approximately.

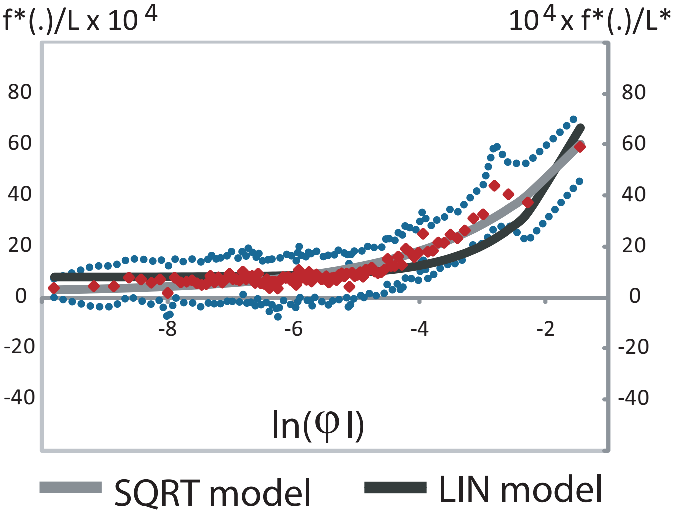
Future research:

- Checking the validity of invariance predictions in other samples,
- Improving the accuracy of estimates and the triangulation of proportionality constants.

Invariant Log-Normality of Portfolio Transition Order Size

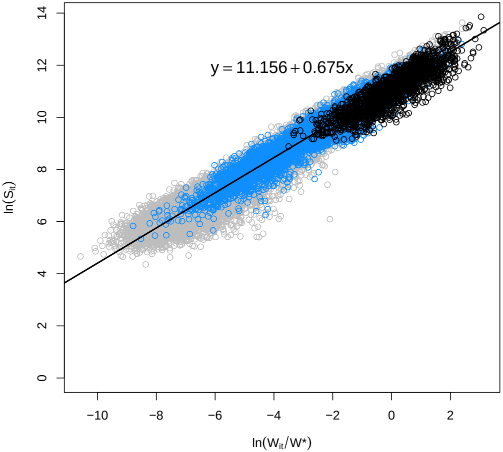


Linear versus Square Root Model

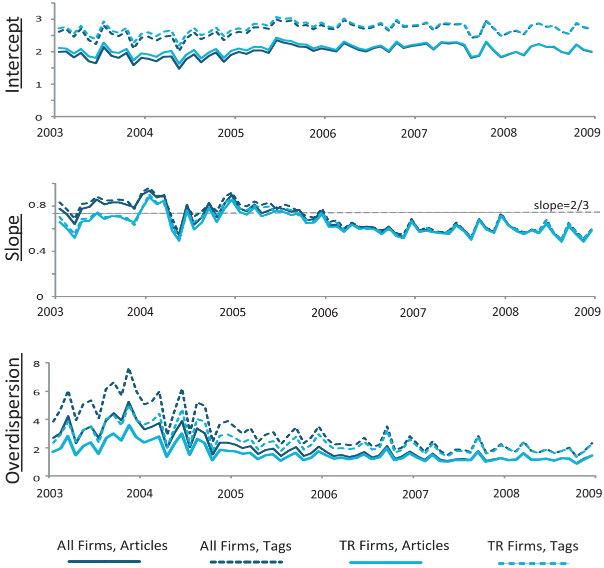


Switching Points: Korean Data

The fitted line for the regression of the number of switching points on trading activity is $\ln(S_{it}) = 11.156 + \mathbf{0.675} \cdot \ln(W_{it}/W^*)$. The invariance-implied slope is **2/3**.



News Articles



NYSE TAQ Prints, 1993